ST. HENRY'S COLLEGE KITOVU

A'LEVEL PURE MATHEMATICS P425/1 SEMINAR QUESTIONS 2019

ALGEBRA

1. Solve the simultaneous equations:
   (a) $x^2 + y^2 = 5$, $\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4}$
   (b) $\frac{x}{y} + \frac{y}{x} = \frac{17}{4}$, $x^2 - 4xy + y^2 = 1$

2. Find the range of values of x for which
   (a) $\frac{2x+1}{x+2} > \frac{1}{2}$.
   (b) $|2x+1| > 7$

3. Resolve into partial fractions
   (a) $\frac{x^3 + x^2 + 4x}{x^2 + x - 2}$
   (b) $\frac{3x^2 + 8x + 13}{(x-1)(x^2 + 2x + 5)}$
   (c) $\frac{2x^3 + 2x^2 + 2}{(x+1)^2(x^2+1)}$

4. Solve the following equations:
   (a) $2^{3x+1} = 5^{x+1}$
   (b) $9^x - 4(3^x) + 3 = 0$
   (c) $\log_x 9 + \log_{x^2} 3 = 2.5$
   (d) $\sqrt{2x-1} - \sqrt{x-1} = 1$
   (e) $\begin{aligned} 2x + 3y + 4z &= 8 \\ 3x - 2y - 3z &= -2 \\ 5x + 4y + 2z &= 3 \end{aligned}$

5. Find:
   (a) The three numbers in arithmetic progression such that their sum is 27 and their product is 504
   (b) The three numbers in a geometrical progression such that their sum 39 and their product is 729.
   (c) The sum of the last three terms of a geometrical progression having n terms is 1024 times the sum of the first three terms of the progression. If the third term is 5, find the last term.

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(d) Prove by induction that $1^3+2^3+....+n^3=\frac{1}{4}n^2(n+1)^2$ and deduce that

$1^3+3^3+5^3....+(2n+1)^3=(n+1)^2(2n^2+4n+1)$

6. Expand:

(a) $\frac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x as far as the term in x^4.

(b) $\left(1-\frac{3}{2}x-x^2\right)^5$ in ascending powers of x as far as the term in x^4.

(c) Find the term independent of x in the expansion of $\left(2x+\frac{1}{x^2}\right)^{12}$ in descending powers of x and find the greatest term in the expansion when $x=\frac{2}{3}$.

(d) Find by binomial theorem, the coefficient of x^8 in the expansion $(3-5x^2)^{1/2}$ in ascending powers of x.

(e) In the binomial expansion of $(1+x)^{n+1}$, n being an integer greater than two, the coefficient of x^4 is six times the coefficient of x^2 in the expansion $(1+x)^{n-1}$. Determine the value of n.

7. (a) Without using the calculator, simplify $\frac{\left(\cos\left(\frac{\pi}{9}\right)+i\sin\left(\frac{\pi}{9}\right)\right)^4}{\left(\cos\left(\frac{\pi}{9}\right)-i\sin\left(\frac{\pi}{9}\right)\right)^5}$

(b) In a quadratic equation $z^2+(p+iq)z+3i=0$. p and q are real constants. Given that the sum of the squares of the roots is 8. Find all possible pairs of values of p and q.

8. (a) How many different arrangements of letters can be made by using all the letters in the word contact? In how many of these arrangements are the vowels separated?
(b) In how many ways can a team of eleven be picked from fifteen possible players.

9. (a) If α and β are the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$.

(b) If α and β are the roots of the equation $x^2+bx+c=0$, form the equation whose roots are $\frac{1}{\beta^3}$ and $\frac{1}{\alpha^3}$. If in the equation above $\alpha\beta^2=1$, prove that $a^3+c^3+abc=0$

10. (a) If $z=x+iy$ and $\bar{z}$ is the conjugate of z, find the values of x and y such that $\frac{1}{z}+\frac{2}{\bar{z}}=1+i$

(b) If x, y, a and b are real numbers and if $x+iy=\frac{a}{b+\cos\theta+i\sin\theta}$. Show that

$(b^2-1)(x^2+y^2)+a^2=2abx$.

(c) If n is an integer and $z=cos\theta+sin\theta$, show that $2\cos n\theta=z^n+\frac{1}{z^n}$, $2i\sin n\theta=z^n-\frac{1}{z^n}$.

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Use the result to establish the formula $8cos^4\theta = cos4\theta + 4cos2\theta + 3$.

(e) If z is a complex number and $\left|\frac{z-1}{z+1}\right| = 2$, find the equation of the curve in the Argand diagram on which the point representing z lies.

TRIGONOMETRY

11. If $sin\theta + sin\beta = a$ and $cos\theta + cos\beta = b$, show that $\cos^2\left(\frac{\theta-\beta}{2}\right) = \frac{1}{4}\left(a^2+b^2\right)$

12. Show that $\sin 7x + \sin x - 2\sin 2x \cos 3x = 4\cos^3 3x$

13. If A, B and C are angles of a triangle, show that:

(i) $\cos A + \cos(B-C) = 2\sin B \sin C$

(ii) $\cos\frac{C}{2} + \sin\frac{A-B}{2} = 2\sin\frac{A}{2}\cos\frac{B}{2}$

14. Express $y = 8\cos x + 6\sin x$ in form of $R\cos(x-\alpha)$ where R is positive and α is acute. Hence find the maximum and minimum values of $\frac{1}{8\cos x + 6\sin x + 15}$ and the corresponding angle respectively.

15. Show that:

(a) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(b) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(c) Find x if $\tan^{-1} x + \tan^{-1}(1-x) = \tan^{-1}\left(\frac{4}{3}\right)$

16. (a) Show that $\cos 4\theta = \frac{\tan^4\theta - 6\tan^2\theta + 1}{\tan^4\theta + 2\tan^2\theta + 1}$

(b) Solve the equation $8\cos^4 x - 10\cos^2 x + 2 = 0$ for x in the range of $0° \le x \le 180°$

17. (a) If $\tan\theta = \frac{1}{p}$ and $\tan\beta = \frac{1}{q}$ and $pq = 2\text{p}$, show that $\tan(\theta+\beta) = p+q$

(b) Show that $\sin 2A + \cos 2A = \frac{(1+\tan A)^2 - 2\tan^2 A}{1+\tan^2 A}$

18. If α, β and γ are all greater than $\frac{\pi}{2}$ and less than 2π and $\sin\alpha = \frac{1}{2}$, $\tan\beta = \sqrt{3}$, $\cos\gamma = \frac{1}{\sqrt{2}}$.

Find the value of $\tan(\alpha+\beta+\gamma)$ in surd form.

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19. Solve for x in the range $0°$ to $360°$

(a) $3\cos^2 x - 3\sin x\cos x + 2\sin^2 x = 1$

(b) $4\cos x = 3\tan x + 3\sec x$

20. Prove that $4\cos\theta\cos 3\theta + 1 = \frac{\sin 5\theta}{\sin\theta}$. Hence find all the values of θ in the range $0°$ to $180°$ for which $\cos\theta\cos 3\theta = \frac{-1}{2}$

VECTORS

21. The coordinates of the points A and B are (0,2,5) and (-1,3,1) and the equation of the line L is $\frac{x-3}{2} = \frac{y-2}{-2} = \frac{z-2}{-1}$

(i) Find the equation of the plane containing the point A and perpendicular to L and verify that B lies in the plane.

(ii) Show that the point C in which L meets the plane is (1,4,3) and find the angle between CA and CB

22. (a) A body moves such that its position is given by $OP = (3sint)i + (3cost)j$ where O is the origin and t is the time. Prove that the velocity of the particle when at P is perpendicular to OP.

(b) The lines L_1 and L_2 have Cartesian equations $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-6}{1}$.

Show that L_1 and L_2 intersect and find the coordinates of the point of intersection.

23. (a) Find the acute angle between the lines whose equations are $\frac{x-2}{-4} = \frac{y-3}{3} = \frac{z+1}{-1}$ and $\frac{x-3}{2} = \frac{y-1}{6} = \frac{z+1}{-5}$.

(b) The points A and B have coordinates (1,2,3) and (4,6,-2) respectively and the plane has equation $x + y - z = 24$. Determine the equation of the line AB, hence the angle this line makes with the plane.

24. (a) Find the perpendicular distance of the line $\frac{x-5}{1} = \frac{y-6}{2} = \frac{z-3}{4}$ from the point (-6,-4,-5).

(b) Find the shortest distance between the two skew lines $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z-3}{1}$ and $\frac{x}{2} = \frac{y+1}{1} = \frac{z-1}{3}$ respectively.

(c) Find the perpendicular distance of the plane $2x - 14z + 5z = 10$ from the origin.

25. (a) Show that the line $\frac{x-2}{2} = \frac{y-2}{-1} = \frac{z-3}{3}$ is parallel to the plane$4x - y - 3z = 4$ and find the perpendicular distance from the line to the plane.

(b) Find the Cartesian equation of the line of intersection of the two planes $2x - 3y - z = 1$ and $3x + 4y + 2z = 3$.

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26. (a) Find the Cartesian equation of the plane containing the point (1,3,1) and parallel to the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$
(b) Find the Cartesian equation of the plane containing the points (1,2,-1), (2,1,2) and (3,-3,3).

27. Given the points A, B and C with coordinates (2,5,-1), (3,-4,2) and C(-1,2,1). Show that ABC is a triangle and find the area of the triangle ABC

28. (a) Find the angle between the parallel planes $3x + 2y - z = -4$ and $6x + 4y - 2z = 6$.
(b) Find the acute angle between the planes $2x + y + 3z = 5$ and $2x + 3y + z = 7$

29. The points A and B have coordinates (2,1,1) and (0,5,3) respectively. Find the equation of the line AB. If C is the point (5,-4,2). Find the coordinates of D on AB such that CD is perpendicular to AB. Find the equation of the plane containing AB and perpendicular to the line CD.

30. (a) Given that $OP = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ and $OQ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, find the coordinates of the point R such that $\overline{PR} = \overline{PQ} = 1:2$ and the points P, Q and R are collinear.
(b) A and B are the points (3,1,1) and (5,2,3) respectively, and C is a point on the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. If angle BAC=$90^0$, find the coordinates of C

ANALYSIS

31. Differentiate from first principles
(a) $y = \tan^{-1} x$
(b) $y = ax^n$
(c) $y = \sin 3x$

32. Find the derivative of:
(a) $y = 5\sin^{-1}(4x)$
(b) $y = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$
(c) $y = \frac{\sin x}{x^2 + \cos x}$
(d) $y = \sqrt{\frac{x}{1+x}}$

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33. Find:

(a) $\int \sin^{-1} x$

(b) $\int \frac{dx}{x^2+4x+13}$

(c) $\int \frac{dx}{x\log_e x}$

(d) $\int \frac{dx}{(1+x^2)\tan^{-1} x}$

(e) Show that $\int_0^2 \sqrt{\frac{x}{4-x}}dx = \pi - 2$

(f) Show that $\int_1^{10} x\log_{10} x = 50 - \frac{99}{4\ln 10}$

34. (a) If $x = t^3$ and $y = 2t^2$. Find $\frac{dy}{dx}$ in terms of t and show that when $\frac{dy}{dx} = 1$, $x = 2$ or $x = \frac{10}{27}$

(b) If $y = \frac{2t}{1+t^2}$ and $x = \frac{1-t^2}{1+t^2}$, find $\frac{d^2y}{dx^2}$ in terms of t

35. Given that:

(a) $y = \sqrt{4+3\sin x}$, show that $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 4$

(b) $y = e^{2x}\cos 3x$, show that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$

(c) $y = \left(x + \sqrt{1+x^2}\right)^p$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2 y = 0$

(d) $y = \sin(\log_e x)$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

36. (a) Find the volume generated when the area enclosed by the curve $y = 4x - x^2$ and the line $y = 2x$ is rotated completely about the x – axis.

(b) Find the area contained between the two parabolas $4y = x^2$ and $4x = y^2$.

(c) Find the area between the curve $y = x^3$, the x – axis and the lines $y = 1, y = 8$.

(d) Find the area of the curve $x^2 + 3xy + 3y^2 = 1$

(e) Show that in the solid generated by the revolution of the rectangular hyperbola $x^2 - y^2 = a^2$ about the x – axis, the volume of the segment of height a from the vertex is $\frac{4}{3}\pi a^3$

37. (a) A right circular cone of semi – vertical angle θ is circumscribed about a sphere of radius R. show that the volume of the cone is $V = \frac{1}{3}\pi R^3(1+\operatorname{cosec}\theta)^3 \tan^2\theta$ and find the value of θ when the volume is minimum.

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(b) Water is poured into a vessel, in the shape of a right circular cone of vertical angle 90°, with the axis vertical, at the rate of $125\text{cm}^3/s$. At what rate is the water surface rising when the depth of the water is 10cm?

38. Sketch the curve $y = \dfrac{x}{x+2}$. Find the area enclosed by the curve, the lines $x = 0, x = 1$ and the line $y = 1$. Also find the volume generated when this area revolves through 2π radians about the line $y = 1$.

39. Solve the differential equations below:

(a) $\dfrac{1}{3x}\dfrac{dy}{dx} + \cos^2 y = 1$, when $x = 2$ and $y = \dfrac{\pi}{4}$

(b) $(x-y)\dfrac{dy}{dx} = x + y$, when $x = 4$ and $y = \pi$

(c) $\dfrac{dy}{dx} + 3y = e^{2x}$, when $x = 0$ and $y = \dfrac{6}{5}$

40. In a certain type of chemical reaction a substance A is continuously transformed into a substance B. throughout the reaction, the sum of the masses of A and B remains constant and equal to m. The mass of B present at time t after the commencement of the reaction is denoted by x. At any instant, the rate of increase of mass of B is k times the mass of A where k is a positive constant.

(a) Write down a differential equation relating x and t

(b) Solve this differential equation given that $x = 0$ and $t = 0$. Given also that $x = \dfrac{1}{2}m$ when $t = \ln 2$, determine the value of k and show that at time t, $x = m(1 - e^{-t})$. Hence find:

(i) The value of x (in terms of m) when $t = 3\ln 2$

(ii) The value of t when $x = \dfrac{3}{4}m$

GEOMETRY

41. (a) Find the equation of a line which makes an angle of 150° with the x – axis and y – intercept of -3 units.

(b) Find the acute angle between the lines $3y - x = 4$ and $6y - 3x - 5 = 0$

(c) OA and OB are equal sides of an isosceles triangle lying in the first quadrant. OA and OB make angles θ_1 and θ_2 with x – axis respectively. Show that the gradient of the bisector of the acute angle AOB is $\operatorname{cosec}\theta - \cot\theta$ where $\theta = \theta_1 + \theta_2$

(d) Find the length of the perpendicular from the point P(2,-4) to the line $3x + 2y - 5 = 0$

42. (a) Find the equation of the circle with centre (4,-7) which touches the line $3x + 4y - 9 = 0$

(b) Find the equation of the circle through the points (6,1), (3,2), (2,3)

(c) Find the equation of the circumcircle of the triangle formed by three lines $2y - 9x + 26 = 0$, $9y + 2x + 32 = 0$ and $11y - 7x - 27 = 0$

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43. (a) Find the length of the tangent from the point (5,6) to the circle $x^2 + y^2 + 2x + 4y - 21 = 0$.

(b) Find the equations of the tangents to the circle $x^2 + y^2 = 289$ which are parallel to the line $8x - 15y = 0$

(c) Find the equation of the circle of radius $12\frac{4}{5}$ which touches both the lines $4x - 3y = 0$ and $3x + 4y - 13 = 0$ and intersects the positive y – axis.

(d) A circle touches both the x – axis and the line $4x - 3y + 4 = 0$. Its centre is in the first quadrant and lies on the line $x - y - 1 = 0$. Prove that its equation is $x^2 + y^2 - 6x - 4y + 9 = 0$

44. Find the equations of the parabolas with the following foci and directrices:

(i) Focus (2,1), directrix $x = -3$

(ii) Focus (0,0), directrix $x + y = 4$

(iii) Focus (-2,-3), directrix $3x + 4y - 3 = 0$

45. (a) Show that the curve $x = 5 - 6y + y^2$ represents a parabola. Find its focus and directrix, hence sketch it.

(b) Find the equation of the normal to the curve $y^2 = 4bx$ at the point $P(bp^2, 2bp)$. Given that the normal meets the curve again at $Q(bq^2, 2bq)$, prove that $p^2 + pq + 2 = 0$

46. (a) Show that the equation of the normal with gradient m to the parabola $y^2 = 4ax$ is given by $y = mx - 2am - am^3$.

(b) P and Q are two points on the parabola $y^2 = 4ax$ whose coordinates are $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ respectively. If OP is perpendicular to OQ, show that $pq = -4$ and that the tangents to the curve at P and Q meet on the line $x + 4a = 0$

47. (a) A conic is given by $x = 4\cos\theta$, $y = 3\sin\theta$. Show that the conic is an ellipse and determine its eccentricity

(b) Given that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $c^2 = a^2m^2 + b^2$. Hence determine the equations of the tangents at the point (-3,3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

48. (a) Show that the locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are at right angles to one another is a circle $x^2 + y^2 = a^2 + b^2$.

(b) The normal to the ellipse $x^2 + y^2 = 100$ at the points A(6,4) and B(8,3) meet at N. If P is the mid – point of AB and O is the origin, show that OP is perpendicular to ON.

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49. (a) P is a point $(ap^2, 2ap)$ and Q the point $(aq^2, 2aq)$ on the parabola $y^2 = 4ax$. The tangents at P and Q intersect at R. Show that the area of triangle PQR is $\frac{1}{2}a^2(p-q)^3$

(b) The normal to the parabola $y^2 = 4ax$ at $P(ap^2, 2ap)$ meets the axis of the parabola at M and MP is produced beyond P to Q so that MP = PQ. Show that the locus of Q is $y^2 = 16a(x+2a)$

50. (a) The normal to the rectangular hyperbola $xy = 8$ at the point (4,2) meets the asymptotes at M and N. Find the length of MN

(b) The tangent at P to the rectangular hyperbola $xy = c^2$ meets the lines $x - y = 0$ and $x + y = 0$ at A and B and Δ denotes the area of triangle OAB where O is the origin. The normal at P meets the x – axis at C and the y – axis at D. if Δ_1 denotes the area of the triangle ODC. Show that $\Delta^2\Delta_1 = 8c^6$

END

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